Robust Consensus Design of uncertain Multi Agent Systems with bounded gains and incremental nonlinear interactions

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***Abstract*— This paper presents design conditions to address the robust leader-follower consensus problem in uncertain Multi-Agent Systems (MASs) with bounded gains and nonlinear interactions. The gains are restricted to an interval of values determined by either parameter uncertainties or the constraints of the agents, inherent in an industrial setting. Additionally, we model the unavoidable nonlinear uncertainties in MAS and its communication architecture using incremental sector-bounded time-varying nonlinear functions, which are particularly well-suited for relative state information-based cooperative protocols. By leveraging Lyapunov stability theory and the Linear Matrix Inequality (LMI) technique, we establish LMI conditions for the distributed design of control gains for both leader and followers. The proposed LMIs exhibit scalability, as they remain independent of the number of nodes and do not necessitate global information regarding the Laplacian eigenvalues, making them well-suited for the design of industrial large-scale MASs. Finally, we numerically and experimentally validate the proposed design approach for paradigmatic scenarios: industrial warehouse inventory management systems and leader-follower algorithm for smart grid and decision-making management applications.**

keyword Distributed design, uncertainties, constraints, cyber-physical systems, industrial applications, multi agent systems, LMIs, leader-follower consensus.

I. Introduction

Many industrial and civil cyber-physical systems of practical interest can be modeled as networks and/or Multi Agent Systems (MASs). For such systems, distributed strategies have been proposed to solve *agreement* problems (i.e. consensus/synchronization) when the control is applied to network nodes (leaderless consensus) and/or to a small fraction of agents (leader follower consensus or pinning control, see i.e [1], [2] and references therein). Many works extend the solution of consensus problems to the uncertain MAS scenarios where the system at node presents nonlinear or Lure-like uncertainty (see i.e. [3]–[6] and references therein). The consensus problem can be rephrased as stabilization problem of the system error between the state variable of interest and the agreement equilibrium in order to provide LMI conditions for closed loop network stabilization ([4], [7], [8] [9]–[12], just to cite a few). In [7] authors propose an approach to analyze and design distributed robust consensus control algorithms for MASs subject to relative state constraints, modeled by Lure-like sector-bounded functions. Other approaches provided conditions that require network global information related to the graph Laplacian matrix ([4], [12], [13]) or that could be difficult to test [8].

Recently a class of Lure-like nonlinearity has been considered, named incremental nonlinear sector bounded ([14], [15]), more natural in the context of agreement problems characterized by relative state information-based protocols. In [14] incremental nonlinearities are considered at node and sufficient LMI conditions, independent from the number of agents, are formulated to assess synchronization of undirected networks. One drawback is that they depend on the smallest nonzero and largest Laplacian eigenvalues thus requiring a priori knowledge of the network graph. The framework is extended to direct strongly connected networks in the presence of disturbance in [15]. Therein LMI conditions are proposed for global consensus with the protocol gain depending on the graph algebraic connectivity, which might not be easy to compute. To overcome the above Laplacian-based strategies that require the knowledge of global information about the Laplacian matrix, adaptive laws for gains or coupling strengths, say c_i, adjustment can be devised when c_i are free parameters to be designed (i.e. [16], [9], [11], [17]).

In many real industrial and civil MAS applications the system at node, the gains and the information propagated through the protocol could be subject to given uncertainties, for instance due to distortion of communication channels, model system and actuator parameter mismatches, spread out of agents' uncertainties to their own neighbors through the interaction protocol ([17], [18], [7], [4], [19]). In addition, the coupling strengths could be intrinsically heterogeneous (i.e. due to different sizes of the systems), lower and upper bounded to account for some agent constraints [20] (i.e. for mobile robots, due to minimal inertia and boundedness of control inputs). This restricts the coupling strengths to specified values within a given interval. In such scenarios, c_i are no longer free parameters to be used for the above Laplacian-based strategies where the tuning of c_i is carried out accounting for global information about the network graph. Similarly, the cited adaptive coupling strengths schemes are not easy

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°The results of this work have been developed within the project PRIN 20225MESZB - 'Resilient Optimization in Distributed Cyber-Physical Control Problems Subject to Adversarial Behaviour,' funded by the European Union's NextGenerationEU program.

to apply because they are not originally designed to account for c_i restricted to a given bounded interval: for instance the final value of c_i cannot be determined a priori to belong to a specific constrained, bounded interval.

In this respect, the paper's contributions are:

1) differently from the above frameworks considered in the literature, herein we deal with uncertain multi-agent systems simultaneously characterized by i) bounded gains and nonlinear uncertainties in the agent and information propagated through the consensus protocol. Specifically, the gains c_i are restricted to a given interval of values because either they are subject to bounded uncertainties or they have to cope with specific agent's constraint (i.e. actuator or input saturation). In such scenarios the gains are no longer free parameters and are restricted to a given bound, therefore Laplacian-depending conditions and adaptive strategies cannot be easily applied. The nonlinear uncertainties of MAS and its communication architecture affecting the protocol are modelled by incremental time-varying nonlinear uncertainties;

2) LMI design conditions are formulated to design in a distributed way the free (Leader and followers) protocol gains in order to assess the robust leader-follower consensus. The solution to LMIs is distributed and does not require global information about the Laplacian eigenvalues. It relies only on extremal values related to the degrees and coupling strengths, such as maximum and minimum values, that are more easily determined or estimated in a distributed manner. The proposed LMI conditions are low computationally demanding as their number and size are independent on the network size. This makes them scalable and suitable also for large scale industrial and civil MASs. Differently than passivity based approaches, there is no assumption about the passivity of an isolated node or the network topological symmetry (e.g. the graph can be directed). The rest of paper is outlined as follows. In Sec. II the notation and problem statement are introduced. In Sec. III LMI conditions for robust leader follower consensus design are formulated. Finally application of the design conditions to two representative industrial scenarios is shown in Sec. IV, while conclusions are outlined in Sec. V.

II. Notation and Problem Statement

Throughout the paper $\mathbb{R}^n$ denotes the n dimensional space with real coordinates and $\mathcal{C}^-$ stands for the complex set with nonpositive real parts. All column vectors x are assumed to be real with x_i being the i-component. $(\cdot)^T$, $|(\cdot)|$ and $\|(\cdot)\|$ denote the transpose, the absolute value and the euclidean norm of x, respectively. We use upper letter to denote matrix and write I (resp. 0) for the identity (resp. null) matrix of opportune dimensions. $\|M\|$ is the induced 2-norm of a matrix $M \in \mathbb{R}^{m\times n}$. A square real symmetric matrix A is definite positive (negative) if $x^TAx > 0$ ($x^TAx < 0$) for all non-zero vectors $x \in \mathbb{R}^n$. The notation $A \geq B$ ($\leq$) means $A - B$ is positive (negative) semidefinite matrix. Let a real square matrix A, $A^s = A + A^T$ denotes the double of the symmetric part of A. $\lambda_n(A)$ denotes the eigenvalue of a matrix A with largest real part. For a finite set V, $|V|$ denotes the cardinality of V. Let $G(V, E, A^d)$ be a directed graph (digraph) with the set of nodes $V = \{1, ..., N\}$, set of edges $E \subseteq V \times V$. The network adjacency matrix is denoted by $A^d = [a^d_{ij}]$ with element $a^d_{ij} = 1$ if there is an edge from vertex j (node parent) into vertex i (node child), and $a^d_{ij} = 0$ otherwise. The Laplacian L is the matrix of elements l_{ij} so that $l_{ij} = \sum_{j=1,j\neq i}^N a^d_{ij}$ for $i = j$ and $l_{ij} = -a^d_{ij}$ for $i \neq j$. Clearly L is a zero row sums matrix with non-positive off-diagonal elements. Herein we consider that two nodes i and j are connected if there exists a (directed or undirected) link connecting them, namely $a^d_{ij} + a^d_{ji} > 0$ and we define the set of nodes connected to node i as $D^c_i = \{j \in V : a^d_{ij} + a^d_{ji} > 0\}$. The connectivity degree of node i is defined as $d_i = |D^c_i|$. The set of in-neighbors of the i-th node is defined as $N^i_i = \{j \in V : a^d_{ij} > 0\}$, with connectivity in-degree $d^i_i = |N^i_i|$. A graph G is called undirected if whenever $(n_i, n_j) \in E$, $(n_j, n_i) \in E$ as well. A digraph G is quasi-strongly connected if there exists a node (root) from which any other node is reachable. G has a spanning tree if there exists a spanning tree that is a subgraph of G. Notice that the condition that G has a spanning tree is equivalent to quasi-strongly connectedness. Throughout the paper we consider digraphs with no self-loops or multiple edges. Let the following model of direct MAS composed of N dynamical followers and one leader:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{1}$$

$$\dot{x}_0(t) = Ax_0(t) \tag{2}$$

with $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^q$, $i = 0, \ldots, N$ (0 denotes the leader agent), are the state and the manipulable control input of the i-th follower, respectively. All matrices and parameters are real. Let the network (1), to solve leader-follower consensus problems means to design protocols to asymptotically control the network' followers to reference leader trajectory $x_0(t)$. In order to ensure a finite consensus state are made the standard assumptions:

[A1] The matrix pair (A, B) is stabilizable;

[A2] The digraph contains a direct spanning tree with leader as root;

[A3] $\lambda_n(A) \in \mathcal{C}^-$.

The well known *standard* protocol is considered:

$$u_i = -c_iK\sum_{j=1}^N a^d_{ij}[x_i(t) - x_j(t)] - \delta_ic_iK^L[x_i(t) - x_0(t)],$$

where K and K^L are the protocol gains to be designed, $c_i > 0$ is the node coupling strength between the i-th node and its neighbors. The protocol is distributed as each agent i-th just needs of information received from d^i_i in-neighbor agents and possibly from the leader (if $\delta_i = 1$). However, the standard protocol may no longer hold, as the information generated, exchanged, or received by the agents may be affected by the inherent uncertainties in both the agents and communication channels. These uncertainties capture the intrinsic nonlinearity of agents and variations in signals that occur during data transmission. The latter are caused by factors such as fading, interference, Bit Error Rate, quantization errors, and distortions in terms of amplitude, frequency, and phase, as well as nonlinear behavior in situations with high signal power [21], [22] (i.e. long distance communication as V2V protocols for autonomous driving). Such features, assumed not known exactly, can be well represented by Lure-like, static-

(*) $$M^{pu} = \begin{bmatrix} \frac{(AH)^s + rH + d_M c_M \eta_1 I - 2ab d_m^{pp} \tau I}{d_M^{pp}} & \frac{c_M d_M^{ip}(BT) + c_M BS}{d_m^{pp}} + (a+b)\tau I \\ \frac{c_M d_M^{ip}(BT)^T + c_M (BS)^T}{d_m^{pp}} + (a+b)\tau I & \frac{d_M^{pp} c_M \eta_2 - 2 d_m^{pp} \tau}{d_M^{pp}} I \end{bmatrix}, M^{uu} = \begin{bmatrix} \frac{(AH)^s + rH + d_M^{uu} c_M \eta_1 I - 2ab d_m^{uu} \tau I}{d_M^{uu}} & \frac{c_M d_M^{iu}(BT)}{d_m^{uu}} + (a+b)\tau I \\ \frac{c_M d_M^{iu}(BT)^T}{d_m^{uu}} + (a+b)\tau I & \frac{(d_M^{p} + d_M^{uu}) c_M \eta_2 - 2 d_m^{uu} \tau}{d_M^{uu}} I \end{bmatrix}$$

$\eta_1 = \frac{1}{\sigma^1} + \sigma^2\gamma; \quad \eta_2 = \sigma^1\gamma + \frac{1}{\sigma^2}$ *for any fixed positive constants* σ^1, σ^2

Parameter	**Meaning**	**Parameter**	**Meaning**
d_i^{uu}	connection degree of an unpinned node i and the rest of unpinned nodes	c_M	maximum coupling strength value
$d_M^{uu}(d_m^{uu})$	maximum (minimum) connection degree among unpinned nodes	$d_M(d_m)$	maximum (minimum) connection degree
d_i^{pp}	connection degree of a pinned node i and the rest of pinned nodes	d_M^{iu}	maximum input degree of unpinned nodes
$d_M^{pp}(d_m^{pp})$	maximum (minimum) degree among pinned nodes	d_M^{ip} (d_M^{p})	maximum input (connection) degree of pinned nodes

TABLE I

nonlinear, time-varying, piecewise continuous in t function: $\phi(t,x) : [0,+\infty] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying locally $\forall\ t \geq 0$, $x_1, x_2 \in \mathcal{U} \subseteq \mathbb{R}^n$ (or globally on all $\mathbb{R}^n$) the incremental sector bounded type condition [14], [15]:

$$[z_1 - z_2 - b(x_1(t) - x_2(t))][z_1 - z_2 - a(x_1(t) - x_2(t))] \leq 0 \quad (3)$$

with $b > a \geq 0$, $z_1 = \phi(t, x_1(t))$, $z_2 = \phi(t, x_2(t))$. Therefore, the protocol for each follower i becomes:

$$\begin{aligned} u_i = &- c_i K \sum_{j=1}^{N} a_{ij}^d [\phi(t, x_i(t)) - \phi(t, x_j(t))] \\ &- \delta_i c_i K^L [\phi(t, x_i(t)) - \phi(t, x_0(t))]. \end{aligned} \quad (4)$$

One of the advantages in using (3) lies in the fact that it is not required knowledge of the exact function value of ϕ, but only rely on bounding the function within a certain range or sector of uncertainties (defined by the parameters a and b). This offers a pragmatic approach by allowing control engineers to work with bounded regions of system behavior rather than requiring an in-depth understanding of the exact functional form ϕ. In addition, the ”incremental” feature of the considered sector-bounded functions is particularly suitable for MAS distributed industrial applications where the error in the state between two agents increases with the difference in their state values. This concept applies, for instance, to scenarios like sensor networks, multiple robots, autonomous networks, swarms of drones and autonomous vehicles, where the error increases as the difference in relative position and/or velocity state becomes larger, also due to the disruption caused by the nonideal channels. Therefore the function ϕ accounts for the overall uncertainty that affects the information both generated by a node i-th (e.g. $\phi(t, x_i)$) and propagated through the protocol (4) (e.g. $\phi(t, x_j)$). In this respect ϕ encompasses both time-varying nonlinear uncertainties due to the intrinsic node nonlinearity and disruptions of communication channels.

Moreover, differently most of frameworks considered in the literature, the coupling strength c_i are heterogeneous and restricted to belong a given range of values. Such bound of values of c_i could reflect nodes parameter mismatch, uncertainties and heterogeneity, uncertainties in the designed protocol gains K^L and K due to actuators and protocol implementation in real industrial scenarios, constraints due to physical characteristics of the agent and actuators (i.e. inertia, input saturation). Under this scenario of uncertain MASs, the aim is to solve the leader-followers design problem. For sake of presentation, when no confusion arises, we will omit the explicit dependence of ϕ and x on t. We need to give and recall the following definitions and preliminary results.

Definition 1 (Robust leader follower consensus): The Lure-like uncertain MAS described by (1)-(2)-(4) with bounded heterogeneous coupling strengths c_i, achieves global (resp. local) robust leader follower consensus if the followers' state exponentially converges to the leader trajectory in the presence of uncertainties in (4), globally (resp. locally) fulfilling condition (3).

Problem 2 (Problem Design): Let the MAS described by (1)-(2)-(4) with incremental nonlinear uncertainties (3) in the protocol (4) and bounded heterogeneous coupling strengths c_i, design the control gain K and K^L such that the network achieves robust leader-follower consensus.

Lemma 3: ([23]) The Linear Matrix Inequality (LMI):

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0,$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ is an invertible matrix and $S(x)$ depend affinely on x, is equivalent to the following condition: $R(x) < 0, Q(x) - S(x)R(x)^{-1}S^T(x) < 0$.

Observation 4: ([24]) Let a square negative definite matrix $T \in \mathbb{R}^{n\times n}$, then $N^T T N$ is negative semidefinite for any matrix $N \in \mathbb{R}^{n\times m}$.

Observation 5: ([24]) Given two vectors $u, v \in \mathbb{R}^n$ and a matrix $W \in \mathbb{R}^{n\times n}$, from linear algebra for any $\sigma > 0$ it results: $2u^T W^T v \leq \sigma u^T W^T W u + \dfrac{1}{\sigma} v^T v$.

III. Design conditions for robust leader-follower consensus

Herein we deal with Problem 2 and formulate LMI design conditions to achieve robust leader follower consensus. In the following it is assumed w.l.o.g. that the first $1 \leq l \leq N$ nodes are pinned (or connected to the leader), namely $\delta_i = 1$, $i = 1, ..., l$ and $\delta_i = 0$, $i = l+1, ..., N$ and we will define the set of pinned and unpinned nodes as: $D_p = \{1, 2 \ldots, l\}$ and $D_u = \{l+1, 2, \ldots, N\}$. Moreover for the sake of presentation we introduce notations about connectivity degree among the unpinned and pinned nodes (resp. maximum and minimum) values: $d_i^{uu} = |\{j \in D_u : a_{ij}^d + a_{ji}^d > 0\}|$ for a fixed $i \in D_u$

($d_M^{uu} := \max_i d_i^{uu}$, $d_m^{uu} := \min_i d_i^{uu}$); $d_i^{pp} = |\{j \in D_p : a_{ij}^d + a_{ji}^d > 0\}|$ for a fixed $i \in D_p$, ($d_M^{pp} := \max_i d_i^{pp}$, $d_m^{pp} := \min_i d_i^{pp}$); $d_i^p = |\{j \in D_p : a_{ij}^d + a_{ji}^p > 0\}|$ for a fixed $i \in V$ ($d_M^p := \max_i d_i^p$, $d_m^p := \min_i d_i^u$); $d_i^{iu} = |N_i|$, for a fixed $i \in D_u$ ($d_M^{iu} = max_i d_i^{iu}$); $d_i^{ip} = |N_i|$, for a fixed $i \in D_p$ ($d_M^{ip} = max_i d_i^{ip}$); $d_M = max_i d_i$; $c_M = \max_i c_i$. For the uncommon scenario of single pinned (resp. unpinned) node, the parameters d_m^{pp}, d_M^{pp}, d_M^{ip} and d_M^p (resp. d_m^{uu}, d_M^{uu}, d_M^{iu} and d_M^u) are conventionally assumed equal to 1. In Table I the main parameters used throughout the manuscript are summarised.

In the following, we will present a result addressing the design of K and K^L to robustly stabilize the uncertain MAS.

Theorem 6: Let the closed loop MAS (1)-(2)-(4) and positive scalar r, if there exist positive constants τ and γ, matrices S, T and $H = H^T > 0$ solutions to the following LMIs:

$$M^{pu} < 0 \tag{5}$$

$$M^{uu} < 0 \tag{6}$$

$$(BS)^s \geq 0 \tag{7}$$

$$(BT)^s \geq 0 \tag{8}$$

$$\begin{bmatrix} I & (BT)^T \\ BT & \gamma I \end{bmatrix} \geq 0 \tag{9}$$

with M^{pu} and M^{uu} defined in box **(*)** and all other terms in Table I, then the MAS achieves global (resp. local) robust exponential leader-follower consensus for all nodes initial conditions belong to $\mathbb{R}^n$ (resp. $\mathcal{U}$) if the condition (3) is global (resp. local). The robust global (local) leader-following consensus is achieved with a rate proportional to r and the control gains are designed as: $K^L = SH^{-1}$ and $K = TH^{-1}$.

Proof: We recast the leader-follower consensus problem in terms of stabilization problem by introducing the agreement error $e_i(t) = x_i(t) - x_0(t)$. Being $\sum_{j=1}^N l_{ij}\phi(x_0(t)) = 0$, the equation describing the closed-loop network dynamics under protocol (4) becomes:

$$\begin{aligned} \dot{e}_i(t) &= Ae_i(t) - c_i BK \sum_{j \in N_i} l_{ij}(\phi(x_j) - \phi(x_0)) \\ &- \delta_i c_i BK^L(\phi(x_i) - \phi(x_0)), \quad i = 1, 2, \dots, N \end{aligned} \tag{10}$$

The candidate Lyapunov function $\mathcal{L} = \sum_{i=1}^N e_i^T H^{-1} e_i$ for system (10) is considered, with H symmetric definite positive matrix of opportune dimensions. Then it results:

$$\begin{aligned} &\dot{\mathcal{L}} + r\mathcal{L} = \textstyle\sum_{i=1}^N e_i^T \left(A^T H^{-1} + H^{-1}A\right) e_i \\ &- \textstyle\sum_{i=1}^N \delta_i c_i (\phi(x_i) - \phi(x_0))^T \left(BK^L\right)^T H^{-1} e_i \\ &- \textstyle\sum_{i=1}^N \left(c_i BK \sum_{j=1}^N l_{ij}(\phi(x_j) - \phi(x_0))\right)^T H^{-1} e_i \\ &- \textstyle\sum_{i=1}^N \delta_i c_i e_i^T H^{-1} BK^L(\phi(x_i) - \phi(x_0)) \\ &- \textstyle\sum_{i=1}^N c_i e_i^T H^{-1} BK \sum_{j=1}^N l_{ij}(\phi(x_j) - \phi(x_0)) \\ &+ r \textstyle\sum_{i=1}^N e_i^T H^{-1} e_i \end{aligned}$$

In order to make the LMI linear with respect variables depending on control gains K and K^L, we firstly recast the above derivation as:

$$\begin{aligned} &\dot{\mathcal{L}} + r\mathcal{L} = \textstyle\sum_{i=1}^N e_i^T H^{-1} \left(HA^T + AH\right) H^{-1} e_i \\ &- \textstyle\sum_{i=1}^N \delta_i c_i (\phi(x_i) - \phi(x_0))^T H^{-1} H \left(BK^L\right)^T H^{-1} e_i \\ &- \textstyle\sum_{i=1}^N \left(c_i BKH \sum_{j=1}^N l_{ij} H^{-1}(\phi(x_j) - \phi(x_0))\right)^T H^{-1} e_i \\ &- \textstyle\sum_{i=1}^N \delta_i c_i e_i^T H^{-1} BK^L H H^{-1}(\phi(x_i) - \phi(x_0)) \\ &- \textstyle\sum_{i=1}^N c_i e_i^T H^{-1} BKH \sum_{j=1}^N l_{ij} H^{-1}(\phi(x_j) - \phi(x_0)) \\ &+ r \textstyle\sum_{i=1}^N e_i^T H^{-1} H H^{-1} e_i \end{aligned}$$

Let $S = K^L H$, $T = KH$, the change of variable $\tilde{e}_i = H^{-1}e_i$, $\tilde{\phi}_{i0} = H^{-1}(\phi(x_i) - \phi(x_0))$ for any i, and taking into account the definition of set D_p and D_u, it results:

$$\begin{aligned} &\dot{\mathcal{L}} + r\mathcal{L} = \textstyle\sum_{i \in D_p} \tilde{e}_i^T ((AH)^s + rH)\tilde{e}_i \\ &- \textstyle\sum_{i \in D_p} \tilde{e}_i^T (d_i^i c_i BT + c_i BS)\tilde{\phi}_{i0} - \sum_{i \in D_p} \tilde{\phi}_{i0}^T (d_i^i c_i BT \\ &+ c_i BS)^T \tilde{e}_i + \textstyle\sum_{i \in D_u} \tilde{e}_i^T ((AH)^s + rH)\tilde{e}_i \\ &- \textstyle\sum_{i \in D_u} d_i^i c_i \tilde{e}_i^T (BT)\tilde{\phi}_{i0} - \sum_{i \in D_u} d_i^i c_i \tilde{\phi}_{i0}^T (BT)^T \tilde{e}_i \\ &+ \textstyle\sum_{i=1}^N c_i \left(\sum_{j=1}^N a_{ij}^d BT\tilde{\phi}_{j0}\right)^T \tilde{e}_i \\ &+ \textstyle\sum_{i=1}^N \tilde{e}_i^T c_i \left(\sum_{j=1}^N a_{ij}^d BT\tilde{\phi}_{j0}\right) \end{aligned}$$

From definition of D_p, D_u, D_i^c, d_i^{pp}, d_i^{uu}, the above derivation may be recast as:

$$\begin{aligned} &\dot{\mathcal{L}} + r\mathcal{L} = \sum_{i \in D_p} \textstyle\sum_{j \in D_i^c \bigcap D_p, \, j>i} \Big[\frac{1}{d_i^{pp}}\Big(\tilde{e}_i^T((AH)^s \\ &+ rH)\tilde{e}_i - d_i^i c_i \tilde{e}_i^T (BT)\tilde{\phi}_{i0} \\ &- c_i \tilde{e}_i^T (BS)\tilde{\phi}_{i0} - d_i^i c_i \tilde{\phi}_{i0}^T (BT)^T \tilde{e}_i - c_i \tilde{\phi}_{i0}^T (BS)^T \tilde{e}_i\Big) \\ &+ \tfrac{1}{d_j^{pp}}\Big(\tilde{e}_j^T((AH)^s + rH)\tilde{e}_j - d_j^i c_j \tilde{e}_j^T (BT)\tilde{\phi}_{j0} \\ &- c_j \tilde{e}_j^T (BS)\tilde{\phi}_{j0} - d_j^i c_j \tilde{\phi}_{j0}^T (BT)^T \tilde{e}_j - c_j \tilde{\phi}_{j0}^T (BS)^T \tilde{e}_j\Big)\Big] \\ &+ \sum_{i \in D_p} \textstyle\sum_{j \in D_i^c, \, j>i} \Big(\tilde{e}_i^T \left(c_i a_{ij}^d BT + c_j a_{ji}^d (BT)^T\right)\tilde{\phi}_{j0} + \\ &\tilde{\phi}_{j0}^T \left(c_j a_{ji}^d BT + c_i a_{ij}^d (BT)^T\right)\tilde{e}_i\Big) \\ &+ \sum_{i \in D_u} \textstyle\sum_{j \in D_i^c \bigcap D_u, \, j>i} \Big[\frac{1}{d_i^{uu}}\Big(\tilde{e}_i^T((AH)^s + rH)\tilde{e}_i \\ &- d_i^i c_i \tilde{e}_i^T (BT)\tilde{\phi}_{i0} - d_i^i c_i \tilde{\phi}_{i0}^T (BT)^T \tilde{e}_i\Big) \\ &+ \tfrac{1}{d_j^{uu}}\Big(\tilde{e}_j^T((AH)^s + rH)\tilde{e}_j^T - \\ &d_j^i c_j \tilde{e}_j^T (BT)\tilde{\phi}_{j0} - d_j^i c_j \tilde{\phi}_{j0}^T (BT)^T \tilde{e}_j\Big)\Big] \\ &+ \sum_{i \in D_u} \textstyle\sum_{j \in D_i^c \bigcap D_u, j>i} \Big(\tilde{e}_i^T \left(c_i a_{ij}^d BT + c_j a_{ji}^d (BT)^T\right)\tilde{\phi}_{j0} \\ &+ \tilde{\phi}_{j0}^T \left(c_j a_{ji}^d BT + c_i a_{ij}^d (BT)^T\right)\tilde{e}_i\Big) \end{aligned} \tag{D_1}$$

Now let's consider the following term in the above derivation (D_1):

$$\begin{aligned} \sum_{i \in D_p} \sum_{\substack{j \in D_i^c, \\ j>i}} &\Big(\tilde{e}_i^T \left(c_i a_{ij}^d BT + c_j a_{ji}^d (BT)^T\right)\tilde{\phi}_{j0} \\ &+ \tilde{\phi}_{j0}^T \left(c_j a_{ji}^d BT + c_i a_{ij}^d (BT)^T\right)\tilde{e}_i\Big) \end{aligned} \tag{11}$$

Let any $\sigma_{ij}^1 > 0$ and $\sigma_{ij}^2 > 0$, from Observation 5 the terms in (11) may be bounded as:

$$\begin{aligned} &\tilde{e}_i^T \left(c_i a_{ij}^d BT + c_j a_{ji}^d (BT)^T\right)\tilde{\phi}_{j0} \\ &\leq c_i a_{ij}^d \tfrac{\sigma_{ij}^1}{2} \tilde{\phi}_{j0}^T (BT)^T (BT)\tilde{\phi}_{j0} + c_i a_{ij}^d \tfrac{1}{2\sigma_{ij}^1} \tilde{e}_i^T \tilde{e}_i \\ &+ c_j a_{ji}^d \tfrac{\sigma_{ij}^2}{2} \tilde{e}_i^T (BT)^T (BT)\tilde{e}_i + c_j a_{ji}^d \tfrac{1}{2\sigma_{ij}^2} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} \\ &\leq c_i a_{ij}^d \tfrac{\sigma_{ij}^1}{2} \gamma \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} + c_i a_{ij}^d \tfrac{1}{2\sigma_{ij}^1} \tilde{e}_i^T \tilde{e}_i + c_j a_{ji}^d \tfrac{\sigma_{ij}^2}{2} \gamma \tilde{e}_i^T \tilde{e}_i \\ &+ c_j a_{ji}^d \tfrac{1}{2\sigma_{ij}^2} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0}, \end{aligned}$$

where the last inequality follows from LMI (9). Therefore (11) can be bounded as:

$$\begin{aligned} &\sum_{i \in D_p} \textstyle\sum_{j \in D_i^c, \, j>i} \Big(\tilde{e}_i^T \left(c_i a_{ij}^d BT + c_j a_{ji}^d (BT)^T\right)\tilde{\phi}_{j0} \\ &+ \tilde{\phi}_{j0}^T \left(c_j a_{ji}^d BT + c_i a_{ij}^d (BT)^T\right)\tilde{e}_i\Big) \\ &\leq \sum_{i \in D_p} \textstyle\sum_{j \in D_i^c, \, j>i} \Big(c_i a_{ij}^d \sigma_{ij}^1 \gamma \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} + c_i a_{ij}^d \frac{1}{\sigma_{ij}^1} \tilde{e}_i^T \tilde{e}_i \end{aligned}$$

(**) $$M_{ij}^{pu} = \begin{bmatrix} \frac{(AH)^s+rH+d_M c_M \eta_1 I}{d_i^{pp}} & -\frac{c_i d_i^i (BT)+c_i BS}{d_i^{pp}} & 0 & 0 \\ -\frac{c_i d_i^i (BT)^T + c_i (BS)^T}{d_i^{pp}} & \frac{d_M^p c_M \eta_2 I}{d_i^{pp}} & 0 & 0 \\ 0 & 0 & \frac{(AH)^s+rH+d_M c_M \eta_1 I}{d_j^{pp}} & -\frac{c_j d_i^j (BT)+c_j BS}{d_j^{pp}} \\ 0 & 0 & -\frac{c_j d_i^j (BT)^T + c_j (BS)^T}{d_j^{pp}} & \frac{d_M^p c_M \eta_2 I}{d_j^{pp}} \end{bmatrix}, M_{ij}^{uu} = \begin{bmatrix} \frac{(AH)^s+rH+d_M^{uu} c_M \eta_1 I}{d_i^{uu}} & -\frac{c_i d_i^i (BT)}{d_i^{uu}} & 0 & 0 \\ -\frac{c_i d_i^i (BT)^T}{d_i^{uu}} & \frac{(d_M^p + d_M^{uu}) c_M \eta_2 I}{d_i^{uu}} & 0 & 0 \\ 0 & 0 & \frac{(AH)^s+rH+d_M^{uu} c_M \eta_1 I}{d_j^{uu}} & -\frac{c_j d_i^j (BT)}{d_j^{uu}} \\ 0 & 0 & -\frac{c_j d_i^j (BT)^T}{d_j^{uu}} & \frac{(d_M^p + d_M^{uu}) c_M \eta_2 I}{d_j^{uu}} \end{bmatrix}$$

$$+c_j a_{ji}^d \sigma_{ij}^2 \gamma \tilde{e}_i^T \tilde{e}_i + c_j a_{ji}^d \tfrac{1}{\sigma_{ij}^2} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} \Big) =$$
$$\sum_{i \in D_p} \sum_{j \in D_i^c, \, j>i} \Big(c_i a_{ij}^d \tfrac{1}{\sigma_{ij}^1} \tilde{e}_i^T \tilde{e}_i + c_j a_{ji}^d \sigma_{ij}^2 \gamma \tilde{e}_i^T \tilde{e}_i \Big)$$
$$+ \sum_{i \in D_p} \sum_{j \in D_i^c, \, j>i} \Big(c_i a_{ij}^d \sigma_{ij}^1 \gamma \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} + c_j a_{ji}^d \tfrac{1}{\sigma_{ij}^2} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} \Big)$$
$$\leq \sum_{i \in D_p} c_M d_i \left(\tfrac{1}{\sigma^1} + \sigma^2 \gamma \right) \tilde{e}_i^T \tilde{e}_i + \sum_j c_M d_j^p \left(\sigma^1 \gamma + \tfrac{1}{\sigma^2} \right) \tilde{\phi}_{j0}^T \tilde{\phi}_{j0}$$
$$\leq d_M c_M \eta_1 \sum_{i \in D_p} \tilde{e}_i^T \tilde{e}_i + d_M^p c_M \eta_2 \sum_j \tilde{\phi}_{j0}^T \tilde{\phi}_{j0}, \tag{12}$$

with $\eta_1 = \frac{1}{\sigma^1} + \sigma^2 \gamma$, $\eta_2 = \sigma^1 \gamma + \frac{1}{\sigma^2}$, $c_M = \max_i c_i$, $d_M^p = \max_i d_i^p$, $d_M = \max_i d_i$. The last inequalities follow from the semi-definite positiveness of $\tilde{e}_i^T \tilde{e}_i$ and $\tilde{\phi}_{j0}^T \tilde{\phi}_{j0}$, considering that $\max_{i \in D_p} d_i \leq d_M$ and assuming, for the sake of simplicity, the arbitrary constants $\sigma_{ij}^1 = \sigma^1$, $\sigma_{ij}^2 = \sigma^2$ for all i, j. A similar conclusion can be derived for the last double sum term in derivation (D_1):

$$\sum_{i \in D_u} \sum_{j \in D_i^c \bigcap D_u, \, j>i} \Big(\tilde{e}_i^T \left(c_i a_{ij}^d BT + c_j a_{ji}^d (BT)^T \right) \tilde{\phi}_{j0}$$
$$+ \tilde{\phi}_{j0}^T \left(c_j a_{ji}^d BT + c_i a_{ij}^d (BT)^T \right) \tilde{e}_i^T \Big)$$
$$\leq \sum_{i \in D_u} c_M d_i^{uu} \left(\tfrac{1}{\sigma^1} + \sigma^2 \gamma \right) \tilde{e}_i^T \tilde{e}_i$$
$$+ \sum_{j \in D_u} c_M d_j^{uu} \left(\sigma^1 \gamma + \tfrac{1}{\sigma^2} \right) \tilde{\phi}_{j0}^T \tilde{\phi}_{j0}$$
$$\leq d_M^{uu} c_M \eta_1 \sum_{i_u} \tilde{e}_i^T \tilde{e}_i + d_M^{uu} c_M \eta_2 \sum_{j \in D_u} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} \tag{13}$$

Exploiting bounds (12) and (13) in derivation (D_1) it results:

$$\dot{\mathcal{L}} + r\mathcal{L} \leq \sum_{i \in D_p} \sum_{j \in D_i^c \bigcap D_p, \, j>i} \Big[\tfrac{1}{d_i^{pp}} \Big(\tilde{e}_i^T ((AH)^s + rH) \tilde{e}_i$$
$$- d_i^i c_i \tilde{e}_i^T (BT) \tilde{\phi}_{i0} - c_i \tilde{e}_i^T (BS) \tilde{\phi}_{i0}$$
$$- d_i^i c_i \tilde{\phi}_{i0}^T (BT)^T \tilde{e}_i - c_i \tilde{\phi}_{i0}^T (BS)^T \tilde{e}_i \Big)$$
$$+ \tfrac{1}{d_j^{pp}} \Big(\tilde{e}_j^T (AH)^s + rH) \tilde{e}_j - d_j^i c_j \tilde{e}_j^T (BT) \tilde{\phi}_{j0} - c_j \tilde{e}_j^T (BS) \tilde{\phi}_{j0}$$
$$- d_j^i c_j \tilde{\phi}_{j0}^T (BT)^T \tilde{e}_j - c_j \tilde{\phi}_{j0}^T (BS)^T \tilde{e}_j \Big) \Big] + d_M c_M \eta_1 \sum_{i \in D_p} \tilde{e}_i^T \tilde{e}_i$$
$$+ d_M^p c_M \eta_2 \sum_{j \in D_p} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} + d_M^p c_M \eta_2 \sum_{j \in D_u} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0}$$
$$+ \sum_{i \in D_u} \sum_{j \in D_i^c \bigcap D_u, \, j>i} \Big[\tfrac{1}{d_i^{uu}} \Big(\tilde{e}_i^T ((AH)^s + rH) \tilde{e}_i$$
$$- d_i^i c_i \tilde{e}_i^T (BT) \tilde{\phi}_{i0} - d_i^i c_i \tilde{\phi}_{i0}^T (BT)^T \tilde{e}_i \Big)$$
$$+ \tfrac{1}{d_j^{uu}} \Big(\tilde{e}_j^T ((AH)^s + rH) \tilde{e}_j - d_j^i c_j \tilde{e}_j^T (BT) \tilde{\phi}_{j0}$$
$$- d_j^i c_j \tilde{\phi}_{j0}^T (BT)^T \tilde{e}_j \Big) \Big]$$
$$+ d_M^{uu} c_M \eta_1 \sum_{i \in D_u} \tilde{e}_i^T \tilde{e}_i + d_M^{uu} c_M \eta_2 \sum_{j \in D_u} \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} =$$
$$\sum_{i \in D_p} \sum_{j \in D_i^c \bigcap D_p, \, j>i} \Big[\tfrac{1}{d_i^{pp}} \Big(\tilde{e}_i^T ((AH)^s + rH + d_M c_M \eta_1 I) \tilde{e}_i$$
$$- d_i^i c_i \tilde{e}_i^T (BT) \tilde{\phi}_{i0} - c_i \tilde{e}_i^T (BS) \tilde{\phi}_{i0} - d_i^i c_i \tilde{\phi}_{i0}^T (BT)^T \tilde{e}_i$$
$$- c_i \tilde{\phi}_{i0}^T (BS)^T \tilde{e}_i + d_M^p c_M \eta_2 \tilde{\phi}_{i0}^T \tilde{\phi}_{i0} \Big)$$
$$+ \tfrac{1}{d_j^{pp}} \left(\tilde{e}_j^T (AH)^s + rH + d_M c_M \eta_1 I) \tilde{e}_j \right.$$
$$- d_j^i c_j \tilde{e}_j^T (BT) \tilde{\phi}_{j0} - c_j \tilde{e}_j^T (BS) \tilde{\phi}_{j0} - d_j^i c_j \tilde{\phi}_{j0}^T (BT)^T \tilde{e}_j$$
$$- c_j \tilde{\phi}_{j0}^T (BS)^T \tilde{e}_j + d_M^p c_M \eta_2 \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} \Big) \Big] \tag{S_1}$$
$$+ \sum_{i \in D_u} \sum_{j \in D_i^c \bigcap D_u, \, j>i} \Big[\tfrac{1}{d_i^{uu}} \Big(\tilde{e}_i^T ((AH)^s + rH + d_M^{uu} c_M \eta_1 I) \tilde{e}_i$$
$$- d_i^i c_i \tilde{e}_i^T (BT) \tilde{\phi}_{i0} - d_i^i c_i \tilde{\phi}_{i0}^T (BT)^T \tilde{e}_i + (d_M^p + d_M^{uu}) c_M \eta_2 \tilde{\phi}_{i0}^T \tilde{\phi}_{i0}$$
$$+ \tfrac{1}{d_j^{uu}} \Big(\tilde{e}_j^T ((AH)^s + rH + d_M^{uu} c_M \eta_1 I) \tilde{e}_j^T - d_j^i c_j \tilde{e}_j^T (BT) \tilde{\phi}_{j0}$$
$$- d_j^i c_j \tilde{\phi}_{j0}^T (BT)^T \tilde{e}_j + (d_M^p + d_M^{uu}) c_M \eta_2 \tilde{\phi}_{j0}^T \tilde{\phi}_{j0} \Big) \Big] \tag{S_2}$$

Let $\tilde{x}_i = [\tilde{e}_i^T \quad \tilde{\phi}_{i0}^T]^T$, $\tilde{x}_{ij} = [\tilde{x}_i^T \quad \tilde{x}_j^T]^T$, the double sum term (S_1) in the above derivation may be recast as the quadratic form: $\sum\limits_{i \in D_p} \sum\limits_{j \in D_i^c \bigcap D_p, j>i} \tilde{x}_{ij}^T M_{ij}^{pu} \tilde{x}_{ij}$ where M_{ij}^{pu} is defined by $(^{**})$ (defined on the top of the page). Similar recasting may be carried out for the double sum term (S_2) with M_{ij}^{uu} defined by $(^{**})$. Hence, let $\tilde{\mathcal{L}} = \dot{\mathcal{L}} + r\mathcal{L}$, it is derived:

$$\tilde{\mathcal{L}} \leq \sum_{i \in D_p} \sum_{j \in D_i^c \bigcap D_p, \, j>i} \tilde{x}_{ij}^T M_{ij}^{pu} \tilde{x}_{ij} + \sum_{i \in D_u} \sum_{j \in D_i^c \bigcap D_u, \, j>i} \tilde{x}_{ij}^T M_{ij}^{uu} \tilde{x}_{ij} \tag{14}$$

Additionally, conditions (3) applied to $\phi_{i0} = \phi(t, x_i) - \phi(t, x_0) = \phi(t, e_i + x_0) - \phi(t, x_0)$ may be recast for any $t \geq 0$, i, e_i, x_0 as: $a e_i \leq \phi_{i0} \leq b e_i$ ([4], [9], [13]), or equivalently $2[\phi_{i0} - b e_i]^T [\phi_{i0} - a e_i] \leq 0$. The latter is equivalent to require: $\begin{bmatrix} e_i \\ \phi_{i0} \end{bmatrix}^T \Phi \begin{bmatrix} e_i \\ \phi_{i0} \end{bmatrix} \leq 0$, with $\Phi = \begin{bmatrix} 2abI & -(a+b)I \\ -(a+b)I & 2I \end{bmatrix}$. If the above constraints hold from condition (3), then from Observation 4 and definite positiveness of H^{-1} it results: $\begin{bmatrix} e_i \\ \phi_{i0} \end{bmatrix}^T \begin{bmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{bmatrix} \Phi \begin{bmatrix} H^{-1} & 0 \\ 0 & H^{-1} \end{bmatrix} \begin{bmatrix} e_i \\ \phi_{i0} \end{bmatrix} \leq 0$, or equivalently: $\tilde{x}_i^T \Phi \tilde{x}_i \leq 0$, that in turns implies for all i, j:

$$\tilde{x}_{ij}^T \tilde{\Phi} \tilde{x}_{ij} \leq 0 \tag{15}$$

with $\tilde{\Phi} = diag\{\Phi, \Phi\}$ symmetric semidefinite negative block

Definition of Matrices $\tilde{M}_{ij}^{pu}$, $\tilde{M}_i^{pu}$, $\tilde{M}_{ij}^{uu}$, $\tilde{M}_i^{uu}$. Similar definition holds for $\tilde{M}_j^{uu}$ and $\tilde{M}_j^{pu}$

$$\tilde{M}_{ij}^{pu} = M_{ij}^{pu} - \tau\tilde{\Phi} = \begin{bmatrix} \tilde{M}_i^{pu} & 0 \\ 0 & \tilde{M}_j^{pu} \end{bmatrix}, \tilde{M}_i^{pu} = \begin{bmatrix} M_1^{pu} & M_2^{pu} \\ (M_2^{pu})^T & M_3^{pu} \end{bmatrix} = \begin{bmatrix} \frac{(AH)^s+rH+d_M c_M \eta_1 I}{d_i^{pp}} - 2ab\tau I & -\frac{c_i d_i^i (BT)+c_i BS}{d_i^{pp}} + (a+b)\tau I \\ -\frac{c_i d_i^i (BT)^T + c_i (BS)^T}{d_i^{pp}} + (a+b)\tau I & \frac{d_M^p c_M \eta_2 I}{d_i^{pp}} - 2\tau I \end{bmatrix}$$
$$\tilde{M}_{ij}^{uu} = M_{ij}^{uu} - \tau\tilde{\Phi} = \begin{bmatrix} \tilde{M}_i^{uu} & 0 \\ 0 & \tilde{M}_j^{uu} \end{bmatrix}, \tilde{M}_i^{uu} = \begin{bmatrix} M_1^{uu} & M_2^{uu} \\ (M_2^{uu})^T & M_3^{uu} \end{bmatrix} = \begin{bmatrix} \frac{(AH)^s+rH+d_M^{uu} c_M \eta_1 I}{d_i^{uu}} - 2ab\tau I & -\frac{c_i d_i^i (BT)}{d_i^{uu}} + (a+b)\tau I \\ -\frac{c_i d_i^i (BT)^T}{d_i^{uu}} + (a+b)\tau I & \frac{(d_M^p + d_M^{uu}) c_M \eta_2 I}{d_i^{uu}} - 2\tau I \end{bmatrix}$$

matrix of opportune dimension. The exponential convergence to the consensus equilibrium at rate depending on scalar $r > 0$ is guaranteed if $\tilde{\mathcal{L}} = \dot{\mathcal{L}} + r\mathcal{L} \leq 0$. Applying the S-Procedure ([23]), the latter inequality is assessed accounting for constraints (15) by requiring:

$$\tilde{\mathcal{L}} \leq \tau \sum_{i \in D_p} \sum_{\substack{j \in D_i^c \bigcap D_p, \\ j > i}} \tilde{x}_{ij}^T \tilde{\Phi} \tilde{x}_{ij} + \tau \sum_{i \in D_u} \sum_{\substack{j \in D_i^c \bigcap D_u, \\ j > i}} \tilde{x}_{ij}^T \tilde{\Phi} \tilde{x}_{ij},$$

with $\tau > 0$ is a scalar variable to be determined. From (14) and (15) the latter inequality is guaranteed if the following holds:

$$\sum_{i \in D_p} \sum_{\substack{j \in D_i^c \bigcap D_p, \\ j > i}} \tilde{x}_{ij}^T \tilde{M}_{ij}^{pu} \tilde{x}_{ij} + \sum_{i \in D_u} \sum_{\substack{j \in D_i^c \bigcap D_u, \\ j > i}} \tilde{x}_{ij}^T \tilde{M}_{ij}^{uu} \tilde{x}_{ij} \leq 0, \tag{16}$$

where matrices $\tilde{M}_{ij}^{pu}$ and $\tilde{M}_{ij}^{uu}$ are defined on the bottom of the previous page. The inequality (16) is fulfilled if matrices $\tilde{M}_{ij}^{pu}$ and $\tilde{M}_{ij}^{uu}$ are definite negative for all i, j. This is equivalent to require the definite negativeness of matrices $\tilde{M}_i^{pu}$, $\tilde{M}_j^{pu}$,$\tilde{M}_i^{uu}$ and $\tilde{M}_j^{uu}$. The aim is to prove that the latter condition is guaranteed if LMIs (5) and (6) hold.

Let the sub-matrix $\tilde{M}_i^{uu}$ of $\tilde{M}_{ij}^{uu}$ as defined on the bottom of the previous page. Firstly observe that $(M_3^{uu})^{-1} = \frac{d_i^{uu}}{(d_M^p + d_M^{uu}) c_M \eta_2 - 2 d_i^{uu} \tau} I = m_3 I < 0$ is a diagonal matrix that is definite negative from LMI (6). Indeed a necessary condition for LMI (6) to hold is $(d_M^p + d_M^{uu}) c_M \eta_2 - 2 d_m^{uu} \tau < 0$ being $d_M^{uu} > 0$. Therefore from $0 < d_m^{uu} \leq d_i^{uu}$ it results: $(d_M^p + d_M^{uu}) c_M \eta_2 - 2 d_i^{uu} \tau < (d_M^p + d_M^{uu}) c_M \eta_2 - 2 d_m^{uu} \tau < 0$ that yields $m_3 < 0$. Hence, from the Schur complement (Lemma 3) the condition $\tilde{M}_i^{uu} < 0$ is equivalent to the following:

$$M_{Schur} = M_1^{uu} - (M_2^{uu})(M_3^{uu})^{-1}(M_2^{uu})^T < 0$$

Notice that $M_1^{uu} = \frac{(AH)^s + rH + d_M^{uu} c_M \eta_1 I}{d_i^{uu}} - 2ab\tau I = \frac{(AH)^s + rH - 2ab d_i^{uu} \tau I + d_M^{uu} c_M \eta_1 I}{d_i^{uu}} \leq \frac{(AH)^s + rH - 2ab d_m^{uu} \tau I + d_M^{uu} c_M \eta_1 I}{d_M^{uu}} < 0$ where the inequalities derive from definite positiveness of τI, condition LMI (6), and being $d_i^{uu} \leq d_M^{uu}$, $d_m^{uu} \leq d_i^{uu}$. Additionally, from Observation (4) it results: $-(M_2^{uu})(M_3^{uu})^{-1}(M_2^{uu})^T \geq 0$.

Therefore such term may be bounded as: $-(M_2^{uu})(M_3^{uu})^{-1}(M_2^{uu})^T = |m_3|(-\frac{c_i d_i^i (BT)}{d_i} + (a+b)\tau I)(-\frac{c_i d_i^i (BT)^T}{d_i} + (a+b)\tau I)^T \leq |m_3|\big(\frac{c_i d_i^i (BT)^T}{d_i} \frac{c_i d_i^i (BT)}{d_i} + (a+b)\tau I (a+b)\tau I - (a+b)\tau \frac{c_i d_i^i}{d_i} (BT)^s\big) \leq |m_3|\big(\frac{c_M d_M^{iu} (BT)^T}{d_m^{uu}} \frac{c_M d_M^{iu} (BT)}{d_m^{uu}} + (a+b)\tau I (a+b)\tau I + (a+b)\tau \frac{c_M d_M^{iu}}{d_m^{uu}} (BT)^s\big) \leq |\hat{m}_3|\big(\frac{c_M d_M^{iu} (BT)^T}{d_m^{uu}} \frac{c_M d_M^{iu} (BT)}{d_m^{uu}} + (a+b)\tau I (a+b)\tau I + (a+b)\tau \frac{c_M d_M^{iu}}{d_m^{uu}} (BT)^s\big)$, with $\hat{m}_3 = \frac{d_M^{uu}}{(d_M^{uu} + d_M^{pp}) c_M \eta_2 - 2\tau d_m^{uu}}$.

The derivation follows considering that the terms composing the last inequality are semidefinite positive from construction and if LMI (8) holds. Therefore the following bound can be derived:

$$M_{Schur} \leq \frac{(AH)^s + rH - 2ab d_m^{uu} \tau I + d_M^{uu} c_M \eta_1 I}{d_M^{uu}} + |\hat{m}_3| \cdot$$
$$\cdot \big(\frac{c_M d_M^{iu} (BT)^T}{d_m^{uu}} \frac{c_M d_M^{iu} (BT)}{d_m^{uu}} + (a+b)\tau \frac{c_M d_M^{iu}}{d_m^{uu}} (BT)^s\big) \leq$$
$$\frac{(AH)^s + rH - 2ab d_m^{uu} \tau I + d_M^{uu} c_M \eta_1 I}{d_M^{uu}} - \Big(-\frac{c_M d_M^{iu} (BT)}{d_m^{uu}}$$
$$+ (a+b)\tau I\Big) \hat{m}_3 I \Big(-\frac{c_M d_m^{iu} (BT)^T}{d_m^{uu}} + (a+b)\tau I\Big)^T$$

By the application of Schur complement, the latter term is negative definite if LMIs (6)-(7)-(8) hold, thus implying the definite negativeness of matrices $\tilde{M}_i^{uu}$. Similar argumentations hold for matrices $\tilde{M}_j^{uu}$, thus assessing the definite negativeness of $\tilde{M}_{ij}^{uu}$. Therefore If LMI (6)-(7)-(8) hold, then $\tilde{M}_{ij}^{uu} < 0$ for all i, j. By the same procedure it can be derived that $\tilde{M}_{ij}^{pu} < 0$ if LMI (5)-(7)-(8) hold. From (16) it follows that $\tilde{\mathcal{L}} = \dot{\mathcal{L}} + \tau\mathcal{L} < 0$, namely, the exponential asymptotic stability of the closed loop system (10) towards the zero equilibrium (e.g. robust leader-follower consensus of the original MAS). ■

Remark 7: Theorem 6 addresses Problem 2 through the solution of LMIs (5), (6), and (9) that have a size of $2n \times 2n$, and LMIs (7) through (8) of size $n \times n$. Importantly, the number and size of these LMIs do not depend on the network size N, thus ensuring that the approach is scalable and well-suited for large-scale MAS applications. Furthermore, it's worth noting that these LMIs are distributed in nature, eliminating the need for global information regarding the Laplacian eigenvalues. Instead, they rely on features, such as maximum/minimum degree and coupling strengths, which are more easily measured or estimated in a distributed manner, similar to other approaches discussed in the literature i.e. [7]

Remark 8: The stabilizability condition [A1], in conjunction with the form of the LMIs (5)-(6), make the feasibility of LMIs suitable. For instance, being $(AH)^s + rH - 2d_m^{pp} ab\tau I + d_M c_M \eta_1 I$ and $(d_M^{pp} c_M \eta_2 - 2 d_m^{pp} ab\tau) I$ symmetric matrices, solvibility of LMI (5) may be obtained by the existence of sufficiently large $\tau > 0$ to make the above terms strictly negative diagonal dominant matrix.

Now we extend the approach to the case where the control gains K and K_L and the followers' manipulable controls u_i are constrained. Namely, let the positive upper bounds $\hat{k}$, $\hat{k}_L$ and $\hat{u}_i$, $i \in V$, the following constraints have to be fulfilled; $\|K\| \leq \hat{k}$ and $\|K_L\| \leq \hat{k}_L$, $u_i \leq \hat{u}_i$.
We make the following assumption:

Assumption 9: There exists $\hat{e}$ such that $\|x_i(t) - x_j(t)\| \leq \hat{e}$ $\forall i, j, t \geq 0$.

Corollary 10: Let $\hat{k}$, $\hat{k}_L$, and $\hat{u}_i$ represent the given constraints on K, K_L, and u_i, respectively, and let $\hat{h}$ and r be positive scalars. If there exist positive constants τ and γ, matrices S, T and $H = H^T > 0$ solutions to the LMIs (5)-(9) and to the following:

$$\begin{bmatrix} H & T^T \\ T & \frac{\hat{k}^2}{\hat{h}} I \end{bmatrix} \geq 0; \begin{bmatrix} H & S^T \\ S & \frac{\hat{k}_L^2}{\hat{h}} I \end{bmatrix} \geq 0; \begin{bmatrix} \hat{h} I & I^T \\ I & H \end{bmatrix} \geq 0 \tag{17}$$

then the closed loop MAS (1)-(2)-(4) achieves global (resp. local) robust exponential leader-follower consensus with a rate proportional to r, for all nodes initial conditions belong to $\mathbb{R}^n$ (resp. $\mathcal{U}$) if the condition (3) is global (resp. local). The control gains are designed as $K^L = SH^{-1}$ and $K = TH^{-1}$ and fulfill the given constraints. In addition if Assumption 9 holds and $c_i \leq \frac{\hat{u}_i}{b\hat{e}(d_i \hat{k} + \hat{k}_L)}$ for all $i \in V$, then it results: $u_i \leq \hat{u}_i$.

Proof: If the first LMI of (17) holds, by Lemma 3 it results: $H - \frac{T^T \hat{h} T}{\hat{k}^2} \geq 0$ and hence $\frac{K^T K}{\hat{k}^2/\hat{h}} \leq H^{-1}$. If the last

LMI of (17) holds, it results: $H^{-1} \leq \hat{h}I$. By application of the S-procedure $\frac{K^T K}{\hat{k}^2/\hat{h}} \leq \hat{h}I$, yielding: $\|K\| \leq \hat{k}$. A similar procedure yields to $\|K_L\| \leq \hat{k}_L$. In addition by condition (3) and Assumption 9 it results: $\|\phi(t,x_i) - \phi(t,x_j)\| \leq b\|x_i - x_j\| \leq b\hat{e}$ for all i,j. Hence by protocol (4) we may infer: $\|u_i\| \leq c_i b\hat{e}d_i\|K\| + c_i b\hat{e}\delta_i\|K_L\| = c_i b\hat{e}(d_i\hat{k} + \delta_i\hat{k}_L)$. The required input constraints $\|u_i\| \leq \hat{u}_i$ are fulfilled if it results: $c_i \leq \hat{c}_i = \frac{\hat{u}_i}{b\hat{e}(d_i\hat{k}+\delta_i\hat{k}_L)}$, $\forall i$, and the result follows. ■

IV. Paradigmatic industrial applications

In the following we will validate the proposed approach to paradigmatic industrial applications. LMIs are solved by the LMI-Matlab toolbox.

1) Numerical validation of a Warehouse Inventory Management application: Drones are increasing warehouse inventory management automation, efficiency, and accuracy by providing functions such as inventory tracking and stock replenishment. We consider a swarm of $N = 6$ robots implementing warehouse inventory operations. Each drone is represented by a double-integrator model for each coordinate in the space by using a feedback linearization (see i.e. [25] and references therein): $A = [0 \quad 1; 0 \quad 0] \otimes I_3$, $B = \begin{bmatrix}0 & 1\end{bmatrix}^T \otimes I_3$, $x_i = [p_i^T \quad v_i^T]^T$ with $p_i \in R^3$ and $v_i \in R^3$ representing position and velocity in the space, respectively. The leader dynamic is $\dot{x}_0(t) = Ax_0(t)$ with $x_0(0) = \begin{bmatrix}0 & 0 & 0 & 10 & 10 & 10\end{bmatrix}^T$ while $x_i(0)$, $i = 1,\ldots,6$ are randomly selected. The communication network is direct, with two robots connected to the leader. Both the communications and position/velocity sensors are affected by uncertainties represented by the form $\phi(t,x_i) = x_i(1.5 - 0.5\cos(t))$, $i = 1,\ldots,6$ and bounded according the condition (3) with $a = 1$ and $b = 2$.

Moreover, a realistic scenario involves drones of different masses, with some capable of carrying heavier loads and others suited for lighter items, enabling efficiency handling of diverse inventory items. In addition, drones have different gains representing varying levels of agility and responsiveness. Finally, imposing input control constraints on drones enhances safety, efficiency, and performance, while also compliance with regulations and promoting seamless integration with human operators.

Therefore each robot i-th is affected by uncertain mass [26], denoted as $m_i \in [\check{m}_i, \hat{m}_i]$. These uncertainties can be represented as uncertainties in the corresponding equivalent coupling gains $\tilde{c}_i = \frac{c_i}{m_i}$, which affect the velocity dynamics. Additionally, each robot must satisfy input constraints $\hat{u}_i$, implying upper bounds as derived from Corollary 10: $\tilde{c}_i \leq \hat{c}_i = \frac{\hat{u}_i}{b\hat{e}(d_i\hat{k}+\delta_i\hat{k}_L)}$. Therefore, the bounds of the equivalent gains $\tilde{c}_i$ can be determined based on mass uncertainty and input constraint as $\tilde{c}_i \in [\frac{\bar{c}_i}{\hat{m}_i}, \frac{\bar{c}_i}{\check{m}_i}]$, with $\bar{c}_i > 0$ such that $\frac{\bar{c}_i}{\check{m}_i} \leq \hat{c}_i$. In the considered example, let $\hat{u}_i \in [6, 12]$, $i = 1,\ldots,6$, $\hat{k} = \hat{k}_L = 0.4$, $a = 1$, $b = 2$, $r = 0.01$, $\hat{h} = 1$, $\sigma_1 = 2$, $\sigma_2 = 0.5$, parameters in Table I for the considered topology, and all bounds of $\tilde{c}_i$ falling within the interval $[0.05, 1.5]$, the solution of LMIs (5)-(9) and (17) with respect to γ, τ, S and T yields the design of $K^L = SH^{-1}$ and $K = TH^{-1}$. At the bottom of Fig. 1, the dynamic evolution of the leader tracking error norm $\|e_i\|$ for the robot-followers is depicted, showing robust tracking of the leader's reference trajectory. Meanwhile, the input constraints are satisfied (top of Fig. 1), despite uncertainties in gains and the protocol.

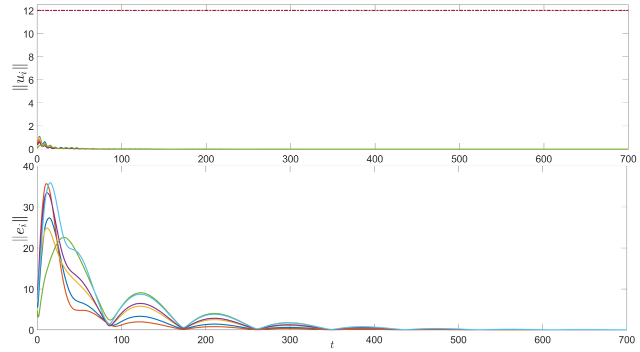

Fig. 1. Bottom: leader tracking error norm $\|\boldsymbol{e_i}\|$, $\boldsymbol{i = 1, \ldots, 6}$; Top: followers' control input norm $\|\boldsymbol{u_i}\|$, $\boldsymbol{i = 1, \ldots, 6}$, (continuous lines) and the maximum constraint value $\boldsymbol{\max_i \hat{u}_i = 12}$ (dashed-dot line)

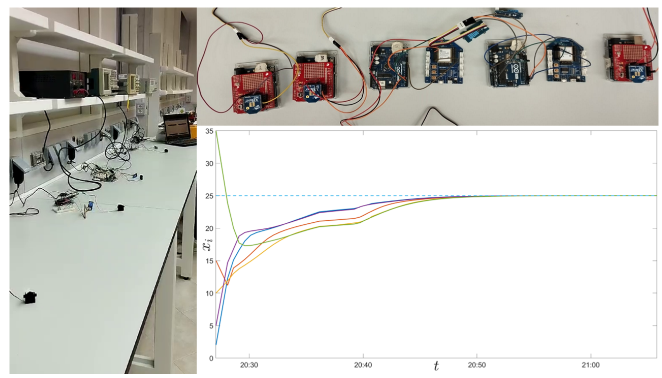

Fig. 2. Top and Left: wireless sensor nodes and experimental set-up; Bottom-right: dynamic evolution of followers' states $\boldsymbol{x_i}$, $\boldsymbol{i = 1, \ldots, 5}$ (continues lines) and leader reference $\boldsymbol{x_0 = 25}$ (dashed line)

2) Experimental validation of a robust distributed algorithm for power grid and decision-making management: Now we experimentally implement a leader-follower consensus-based algorithm, widely used in both power grid management (i.e., [27], [28]) and decision-making for IoT networks [29]. In the first case, the distributed secondary control (DSC) scheme is devised by exchanging information among neighboring Distributed Energy Resources (DERs), to update the nominal voltage and frequency in each primary level control process and further restore the terminal voltage and frequency outputs to their reference values [27], [28]. This is distributively implemented by a leader-follower consensus-based algorithm over a cyber-network where the leader provides the reference value for controllers, and all DERs agree with it.

Similarly, the use of a consensus-based algorithm over a cyber-network is considered in decision-making at IoT edge nodes in service-oriented applications to increase confidence in the measurement data. Moreover, the algorithm can support decision-making in knowledge sharing and integration of multiple services [29]. In the above applications the leader-follower consensus algorithm is distributively carried out over a cyber-network, implemented by a wireless network. In this respect we experimentally validate the proposed approach by designing a robust leader-follower algorithm (4) implemented over a small cyber network composed of 5 off-the shelf wireless sensor nodes (see i.e. top and left section of Fig. 2). The node gains are heterogeneous and uncertain with $c_i \in [2, 4]$. Furthermore, we account for the presence of wireless sensor communication and model/measurement uncertainties as represented in the form of ϕ, bounded by a sector and fulfilling (3) with $a = 1$, $b = 3$ (e.g. $\phi(x_i) = 2x_i + \sin(x_i)$). A ring topology is implemented with a leader reference value $x_0 = 25$. Let $r = 0.05$, $\sigma_1 = 2$, $\sigma_2 = 0.5$, the solutions of LMIs yield: $K^L = 0.34$, $K = 0.2$. At the bottom-right of Fig. 2, the dynamic evolution of followers (continuous line) and leader (dashed-line) reference states shows the achievement of robust leader follower consensus in real experiment, despite the presence of uncertainties.

V. Conclusions

The paper presents LMI conditions for the distributed design of the leader-follower consensus protocol for uncertain Multi-Agent Systems (MASs). Such a framework involves bounded heterogeneous gains (due to uncertainties or input constraints) and communication disruptions, the latter represented by incremental sector-bounded functions. The proposed LMIs are distributed, scalable and computationally efficient, as they do not depend on the number of nodes. Consequently, they are well-suited for handling large-scale industrial and civil MAS applications. The proposed design approach is numerically and experimentally validated in illustrative industrial scenarios.

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